4. The extreme points problem asks whether the convex hull of \( n \) given points in the plane has \( n \) vertices (i.e., whether all of the \( n \) points are “extreme”); note that this is potentially an easier problem than actually computing the convex hull. Model this problem as a set recognition problem, i.e., that of recognizing whether or not an input vector \( x \) belongs to \( W \), for an appropriate set \( W \subseteq \mathbb{R}^2 \). Prove that the number of connected components \( \#W \geq (n-1)! \) and conclude that the algebraic computation tree complexity of the problem is \( \Omega(n \log n) \).

5. Let \( a_1, \ldots, a_k \) and \( b \) be fixed nonzero vectors in \( \mathbb{R}^n \) such that the system of inequalities

\[
\langle a_1, x \rangle \geq 0, \quad \langle a_2, x \rangle \geq 0, \quad \ldots, \quad \langle a_k, x \rangle \geq 0,
\]

in the unknown \( x \in \mathbb{R}^n \), is feasible and implies the inequality \( \langle b, x \rangle \geq 0 \). Here \( \langle , \rangle \) denotes the standard inner product in \( \mathbb{R}^n \). Then it can be shown that \( b \) is a non-negative linear combination of the \( a_i \)'s, i.e., \( b = \sum_{i=1}^{k} \lambda_i a_i \) for some non-negative reals \( \{\lambda_i\} \). This fact is sometimes known as Farkas’s Lemma.

Using Farkas’s Lemma, prove the following two lower bounds in the linear decision tree model (i.e., on input \( x \), each internal node gets to ask a question “\( \sum_{i=1}^{n} c_i x_i \geq 0? \)” where the \( c_i \)'s are constants).

5.1. The complexity of finding the largest of \( n \) given reals is \( n - 1 \).

5.2. The complexity of finding the second largest is at least \( n - 2 + \log n \).

Hint: Once you have solved #5.1, use what you learnt along with a leaf counting argument to solve #5.2.